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On the no-slip boundary condition

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It has been argued that the no-slip boundary condition, applicable when a viscous fluid flows over a solid surface, may be an inevitable consequence of the fact that all such surfaces are, in practice, rough on a microscopic scale: the energy lost through viscous dissipation as a fluid passes over and around these irregularities is sufficient to ensure that it is effectively brought to rest. The present paper analyses the flow over a particularly simple model of such a rough wall to support these physical ideas.

1. Introduction

It is now accepted that, when a viscous fluid flows over a solid surface, there is no relative motion between the fluid and the solid at the interface. Under normal circumstances, the no-slip condition provides a realistic restriction on solutions of the Navier–Stokes equations.† Confidence in the relevance of this boundary condition stems from both direct experimental evidence and the success achieved by theories which incorporate this assumption (see, for example, Goldstein 1938).

The situation is much less clear when we inquire into the origin of the no-slip condition. For gases, kinetic theory and thermodynamic arguments can be invoked to provide an explanation (Lighthill 1963): the gas molecules are required to remain chemically adsorbed onto a solid surface for a time which is sufficiently long to allow a thermal equilibrium to be attained. For liquids, a similar explanation in molecular terms is possible: it can be argued that the intermolecular forces between the liquid and a solid wall give a bond which results in the no-slip condition. However, although this seems plausible in many circumstances, there are situations in which it may be questioned. For example, when the liquid involved does not wet the solid on which it moves, so that the angle of contact between the liquid and solid is close to 180°, then the intermolecular forces giving rise to the above bond may be orders of magnitude smaller than the forces between the liquid molecules themselves. In this situation at least, one wonders whether this explanation can be adequate. Indeed, if the cohesive forces at the interface were entirely absent one would expect a boundary condition of zero shear stress, rather than one of no slip, to be applicable for the motion of the liquid. On this basis, at any actual fluid/solid interface, a condition lying somewhere between these two extremes would seem to be more appropriate.

† Rarefied gases form an exception to this statement. Under certain conditions of low density it is still possible to regard these as continua with Newtonian properties but a slip condition has to be assumed.

A possible alternative explanation of the origin of the no-slip condition, which relies on the fact that all solid surfaces are, in practice, rough on a microscopic scale, has been outlined by Pearson & Petrie (1968). They were forced to consider the boundary conditions applicable at a solid surface in more detail by the observation that a polymer melt does, under certain circumstances, show a slip velocity: while a no-slip condition seems to be relevant for moderate values of the shear rate at the solid boundary, slipping can occur at higher values. This is convincingly shown in experiments performed by Benbow & Lamb (1963), who also offer evidence that this breakdown may sometimes be responsible for the instability termed melt fracture which can occur during an extrusion process. Theoretical work by Pearson & Petrie (1965, 1968) supports this conjecture, for they show that the use of a boundary condition allowing slip can lead to the growth of instabilities in the flow between parallel planes.

In the present paper, the original physical ideas of Pearson & Petrie (1968) are elaborated, and the flow of an incompressible Newtonian fluid over a particular family of models for a rough wall is examined in detail. As remarked above, the actual condition applicable at the interface between a fluid and a solid should lie between the two extremes of zero shear stress and no slip: we here show that, whichever of these two extreme hypotheses we apply on the microscopic scale along the undulations of the wall, the no-slip condition emerges as the relevant macroscopic boundary condition in the sense that deviations from it are of the same order of magnitude as the dimensions of the asperities at the wall. We would therefore expect the same conclusion to hold whatever the actual strength of the bond that exists between the liquid and the solid, so that the no-slip boundary condition as a macroscopic phenomenon can be explained solely in terms of a microscopic roughness of the wall.

For a discussion of the motion near a solid wall, there are three relevant length scales:

(i) l, the length scale associated with the molecular structure of the fluid;

(ii) ϵ, the length scale associated with the roughness of the solid surface;

(iii) L, an appropriate length scale associated with the bulk motion of the fluid. L may be the diameter of a tube through which the fluid passes, the radius of curvature of the surface bounding the flow as it appears on a conventional laboratory scale of measurement, or the thickness of a boundary layer above the solid surface: in the present context we are concerned only with situations where L is much larger than both l and ϵ.

For a typical Newtonian fluid passing over a solid surface we will have $\epsilon \gg l$, so that, when considering the effect of the asperities of the rough surface on the flow, we may still regard the fluid as a continuum. However, for a polymer melt passing over the same surface we may have $\epsilon = O(l)$: the dimensions of the long-chain molecules in the melt will be comparable with those of the asperities, and the motion near the wall cannot be considered solely in terms of the motion of a continuum. Pictorially, the local behaviour at the surface will be more akin to the motion of steel wool over sandpaper, and the macroscopic boundary condition emerging from this interaction might be expected to have more in common with that of solid friction than the no-slip condition usually associated with fluid flow.

Further discussion of the possible behaviour of a polymeric material at a solid boundary may be found in Pearson & Petrie (1968). We here consider in more detail the situation for which $\epsilon \gg l$, and a Newtonian fluid is involved.

In a conventional analysis, a solid wall which, in practice, contains roughness elements with a length scale ϵ is replaced by a smooth boundary at some mean position. The actual flow which occurs in the neighbourhood of this idealized surface is complex, but at distances of order ϵ away from it the flow velocity in the fluid, averaged over a small area parallel to this apparent wall surface, must itself be parallel to the surface. If this average has a magnitude U_s, then U_s would be interpreted on a macroscopic level as a slip velocity. For a Newtonian fluid, with $\epsilon \gg l$, the motion near the wall may be discussed in terms of a continuum model. Furthermore, if the length scales involved are small enough to render negligible any inertial effects in the flow, the equations of Stokes flow will be applicable. No matter what the actual boundary condition at the wall may be, the motion of the fluid over and around the asperities of dimension ϵ must give rise to velocity gradients which are $O(U_s/\epsilon)$ and hence, if μ is the viscosity, to a rate of dissipation of energy which is $O(\mu U_s^2/\epsilon^2)$ per unit volume. If we consider the motion in a region within a distance ϵ of the mean position of the wall, this is a rate of dissipation of energy which is $O(\mu U_s^2/\epsilon)$ per unit area of the apparent wall surface. In the absence of inertial effects, this energy loss must be balanced by the work done by the viscous traction exerted on the fluid in this region by a shear rate which is $O(U_s/\epsilon)$ acting at the edge of the layer near the wall. Thus, the energetics of the flow in the immediate neighbourhood of a rough wall are such that the shear rate κ_w observed at the wall on a macroscopic scale is inevitably $O(U_s/\epsilon)$. Since κ_w must be finite, it follows that $U_s/\kappa_w = O(\epsilon)$, irrespective of the actual boundary condition along the undulations, provided only that this condition does not involve an energy-producing mechanism. In other words, no slip as a macroscopic phenomenon will result from a microscopically rough surface.

To illustrate further the implications of the roughness of a surface, consider the extreme, hypothetical, case where there is no bond between a solid and a fluid in contact, and a zero-shear-stress condition is thus applicable on the microscopic scale. If we apply such a boundary condition for flow between parallel planes which we regard as being perfectly smooth, the result will be a plug flow with a constant velocity across the channel. However, if we apply the same condition on parallel planes which we regard as rough, the above physical arguments imply that the resulting flow will be essentially the parabolic velocity profile with zero slip at the boundary, as sketched in figure 1.

In a general motion with given boundaries we may consider a family of mathematical problems, each characterized by a different value of the roughness parameter ϵ, which gives the size of the asperities on the boundary, these being geometrically similar for different values of ϵ. Some value of ϵ corresponds to the given rough wall, while $\epsilon \to 0$ gives the idealized 'smooth' surface which is normally used for purposes of analysis. One then expects that the velocity field for such a family of problems could be expressed as

$$\mathbf{u}(\mathbf{x},\epsilon) = \mathbf{u}_0(\mathbf{x}) + \epsilon\mathbf{u}_1(\mathbf{x}) + \epsilon^2\mathbf{u}_2(\mathbf{x}) + \dots, \tag{1.1}$$

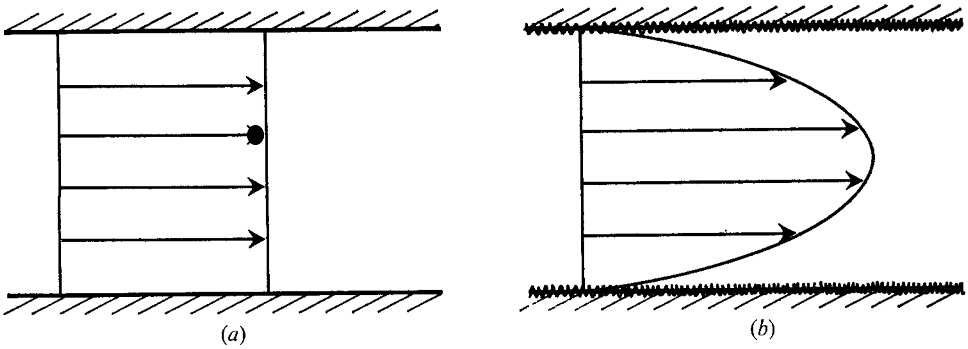

FIGURE 1. Comparison of the expected velocity profiles for channel flow with a zero-shear-stress condition applied at (a) smooth walls and (b) rough walls.

where $\mathbf{x}$ is the position vector. If the true boundary condition along the undulations of the rough wall were known (be it no slip, zero shear stress, or some intermediate condition) one might suppose, *a priori*, that $\mathbf{u}_0(\mathbf{x})$ would be found by applying the same boundary condition at the position of the smoothed surface $\epsilon \rightarrow 0$. However, the above implies that this is not so, and that the expansion forms a singular perturbation in ϵ: whatever the true boundary condition at the rough wall may be, $\mathbf{u}_0(\mathbf{x})$ is determined by applying a no-slip condition at the smoothed wall. The singular nature of the perturbation expansion obviously reflects the singular nature of the bounding surface as $\epsilon \rightarrow 0$: although the limiting surface is continuous and 'looks smooth', mathematically it is nowhere differentiable.

The above arguments have proceeded on the assumption that one length scale, ϵ, suffices to characterize the surface roughness: the surface has irregularities of amplitude a and wavelength λ which are comparable and of order ϵ. The limit envisaged holds a/λ fixed and allows $a \rightarrow 0$. If λ is held fixed while $a \rightarrow 0$ an entirely different behaviour can be expected. In fact, the perturbation expansion will then be regular. Nevertheless, such a calculation does show a behaviour consistent with the above. Nye (1969, 1970), in an entirely different context, considers flow over a rough surface at which a zero-shear-stress condition is applied, taking $a \rightarrow 0$, but λ fixed and finite to effect a linearization. A slip velocity is found which varies as a^{-2}, becoming infinite as $a \rightarrow 0$ but having a finite value for small, but non-zero, amplitudes. Although this is suggestive, we must adopt a quite different approach to tackle the present problem, when both amplitude and wavelength are small and of the same order.

2. Flow near a rough solid surface

We consider the flow in the immediate neighbourhood of a solid surface in a quite general flow. For simplicity the geometry and motion is taken to be locally two-dimensional, i.e. the roughness is assumed to consist of corrugations which are perpendicular to the flow. With this restriction we obtain a tractable problem: we would not expect variations in a third direction to change the results qualitatively.

Dealing with a particular section of the boundary, we introduce an orthogonal, curvilinear co-ordinate system (X, Y) there, with the X axis along the surface and the Y axis perpendicular to it, so that the flow takes place in the X, Y plane. $Y = 0$ is to correspond to some mean smooth curve in the rough surface, and is the flow boundary which would be used in any analysis of the flow on a macroscopic scale.

We now recognize that the surface bounding the flow actually contains roughness elements of length scale ϵ, and to focus attention on these we change variables to

$$x = X/\epsilon, \quad y = Y/\epsilon.$$

To determine the effect of small asperities in the boundary on a given physical situation we now consider the limit $\epsilon \to 0$ with X and Y fixed. In the physical X, Y plane we are considering a succession of problems with a bounding surface which contains asperities of successively smaller size, whilst keeping these irregularities similar in shape. In the x, y plane the asperities have a dimension $O(1)$ and, in the limit $\epsilon \to 0$, the curve $y = 0$ becomes a straight line, so that the (x, y) co-ordinates form a rectangular Cartesian system. For a fixed Y, as $\epsilon \to 0$, then $y \to \infty$, so that the flow in an inner region near the wall covers the upper half of the x, y plane. Suppose that the boundary behaviour of the outer solution – the solution applicable in the main flow – requires there to be a wall slip velocity U_s and a wall shear rate κ_w. The shear rate κ_w is imposed by the outer solution, and it is then required to show that the dynamics in the inner region imply $U_s/\kappa_w = O(\epsilon)$, independently of the actual boundary condition applied at the rough surface.

In the inner flow region we make the lengths non-dimensional using ϵ, the velocities non-dimensional using $\epsilon\kappa_w$ and the pressure non-dimensional using $\mu\kappa_w$. The appropriate limiting form of the Navier–Stokes equation becomes the Stokes equation

$$\nabla p = \nabla^2 \mathbf{u}, \tag{2.1}$$

where ∇ denotes the gradient operator with respect to (x, y), and p and $\mathbf{u}$ are the dimensionless pressure and velocity. The outer boundary condition for this inner region requires the (x, y) components of the dimensionless velocity to vary as

$$(u, v) \sim (U + y, 0) \quad \text{as} \quad y \to \infty, \quad \text{where} \quad U_s = \epsilon\kappa_w U. \tag{2.2}$$

It follows from this that, in the limit,

$$p \to \text{constant} \quad \text{as} \quad y \to \infty. \tag{2.3}$$

This shows the principal difference between the present theory and boundary-layer theory. In the latter case, the pressure in the layer near the wall is independent of y, but its variation along the wall is furnished by the outer solution. Here, the co-ordinate X is scaled as well as the co-ordinate Y so that, although the pressure varies with X in the outer flow, the limits for the inner flow are such that $\partial p/\partial x = O(\epsilon)$ as $y \to \infty$. This fact makes the dominant term of the inner solution largely independent of the outer solution: the latter merely gives a value for κ_w. Here, we do not propose to consider the outer solution, which will depend on the overall geometry of the apparatus, but to regard κ_w as given.

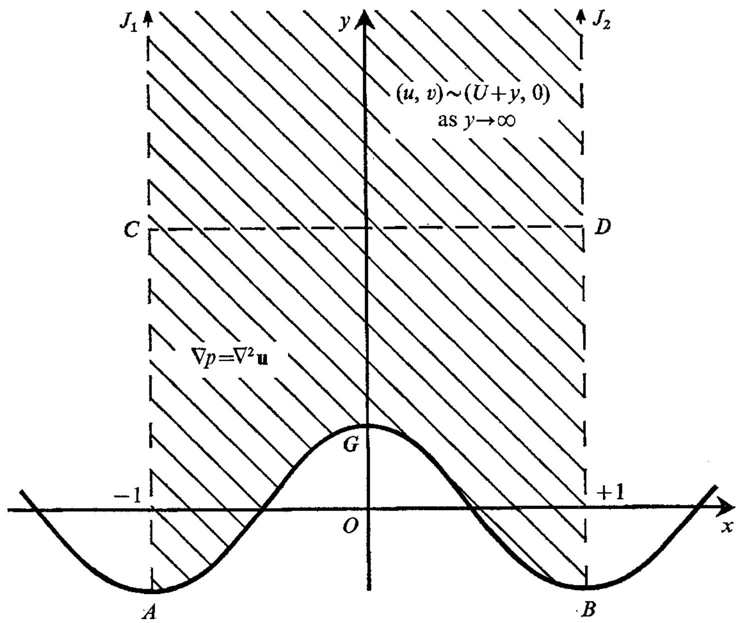

FIGURE 2. Sketch of the flow domain in the x, y plane.

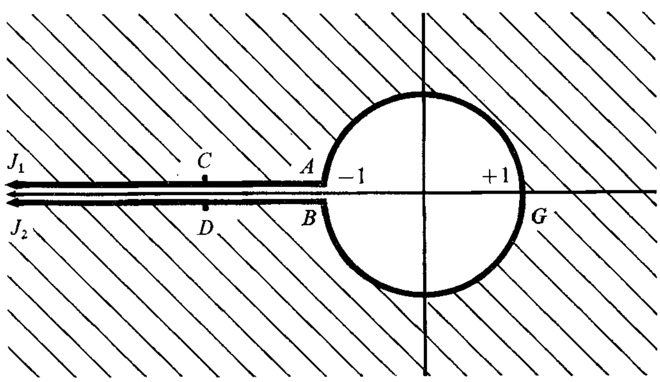

FIGURE 3. Transformed flow domain in the ζ plane.

The problem for the inner region in the limit $\epsilon \to 0$ thus reduces to the solution of the Stokes equation (2.1) in the x, y plane for a unit shear (2.2) over a corrugated plane wall with a length scale which is $O(1)$. It is then required to show that U in (2.2) is determinate and $O(1)$, so that the slip velocity U_s is an $O(\epsilon)$ quantity times κ_w.

3. The boundary shape

If we assume that the undulations are periodic and symmetric, the geometry in the x, y plane may typically be as in figure 2. The velocity, vorticity and pressure at a point on the line AJ_1 are the same as at the point on BJ_2 with the same y co-ordinate. Using this correspondence, it is only necessary to consider the flow in the semi-infinite strip bounded by the lines AJ_1 and BJ_2. With $z = x+iy$, Riemann's theorem implies that there exists a conformal mapping $z = w(\zeta)$ of this strip onto the exterior of the unit circle in the ζ plane, with a cut along the negative real axis. The correspondence of points can be taken as in figure 3, and it is evident from the assumed symmetry that points on AJ_1 and BJ_2 with equal y co-ordinates (e.g. C and D) transform to neighbouring points on either side of the cut. Using square brackets around a quantity to denote the difference between its values at $1+iy$ and at $-1+iy$ (or the difference between the values at the corresponding points in the ζ plane), then $[w(\zeta)] = 2$, whence $[w'(\zeta)] = 0$. It follows that $w'(\zeta)$ is analytic and single-valued in the region exterior to the unit circle, and hence we may write

$$w(\zeta) = \frac{i}{\pi}\log\zeta + iC + \sum_{n=1}^{\infty}\frac{i\alpha_n}{\zeta^n} \quad \text{for} \quad |\zeta| > 1, \tag{3.1}$$

where C and α_n are constants. The imposed symmetry requires all these constants to be real. Writing $\zeta = e^{i\theta}$, the surface is given in parametric form by

$$\left.\begin{aligned} x &= -\frac{\theta}{\pi} + \sum_{n=1}^{\infty}\alpha_n \sin n\theta, \\ y &= C + \sum_{n=1}^{\infty}\alpha_n \cos n\theta, \end{aligned}\right\} \quad \text{for} \quad -\pi \leqslant \theta \leqslant +\pi. \tag{3.2}$$

Choosing C appropriately allows the x axis in figure 2 to be placed at a suitable level, which is to be regarded as the mean position of the rough surface.

While any surface with the requisite periodicity and symmetry can be obtained using a mapping of the form (3.1), we here consider only one particular family of surfaces where the series contains only the first term, i.e. $\alpha_n = 0$ for $n > 1$, while $\alpha_1 = \alpha/\pi$, say. It also proves to be convenient and realistic to take $C = 0$, so that we employ

$$w(\zeta) = \frac{i}{\pi}\log\zeta + \frac{i\alpha}{\pi\zeta}, \tag{3.3}$$

where α is a real parameter. This implies that

$$w'(\zeta) = (i/\pi\zeta^2)(\zeta - \alpha), \tag{3.4}$$

so that α must be restricted by $|\alpha| \leqslant 1$ for the mapping to be conformal for $|\zeta| > 1$. The extreme values of y are $\pm\alpha/\pi$. Thus $|\alpha|/\pi$ is an effective amplitude-to-wavelength ratio, and the choice $C = 0$ has placed the x axis midway between the highest and lowest points of the surface. Moreover, it is easy to see that the effect of replacing α by $-\alpha$ is merely to translate the surface through unit distance in the x direction, so that we may restrict attention to the range $0 \leqslant \alpha \leqslant 1$. The surfaces modelled in the z plane as α varies within this range are sketched in figure 4. $\alpha = 0$ gives the plane surface devoid of any irregularities, while cusps appear on the boundary as $\alpha \to 1$. The flow over this cusped form can be obtained by a limiting process, as it is convenient to assume that $|\alpha| < 1$ during the manipulations. Flows over a more general surface, with more terms in the series

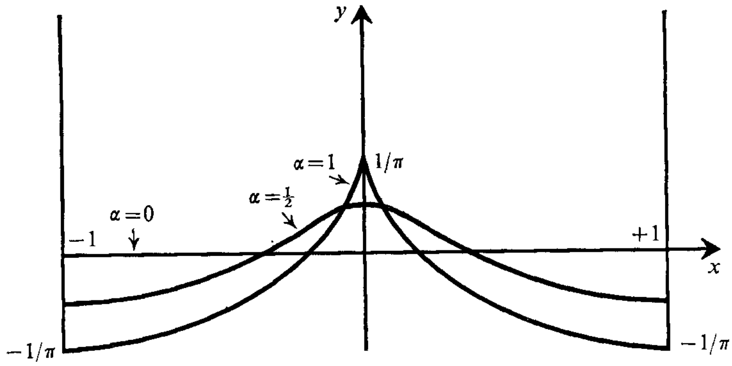

FIGURE 4. Form of the surface modelled by the function $w(\zeta)$ given in (3.3).

of (3.1), can obviously be obtained by methods similar to those to be adopted here, but the amount of labour involved increases with the number of terms employed.

4. Formulation in complex-variable form

Any solution of the Stokes equation (2.1) in two dimensions can be expressed in terms of two functions $\phi(z)$ and $\chi(z)$ which are analytic functions of z within the flow domain. Using the notation of Richardson (1968), we have

$$-v+iu = \phi(z)+z\overline{\phi'(z)}+\overline{\chi'(z)}, \tag{4.1}$$

where an overbar denotes the complex conjugate, while the pressure p and vorticity ω are given by

$$\omega+ip = -4\phi'(z). \tag{4.2}$$

In the present instance it proves more convenient to use, instead of $\chi(z)$, a function $\pi(z)$ defined by

$$\pi(z) = z\phi'(z)+\chi'(z). \tag{4.3}$$

Equation (4.1) then becomes

$$-v+iu = \phi(z)+(z-\bar{z})\overline{\phi'(z)}+\overline{\pi(z)}, \tag{4.4}$$

while the force components $(F_x ds, F_y ds)$ across an element of arc ds are determined by

$$(F_x+iF_y)\,ds = 2d(\phi(z)-(z-\bar{z})\overline{\phi'(x)}-\overline{\pi(z)}). \tag{4.5}$$

A given velocity and stress field determines $\phi(z)$ and $\pi(z)$ up to the addition of a complex constant γ to $\phi(z)$, together with the simultaneous addition of $-\bar{\gamma}$ to $\pi(z)$. If we choose this constant appropriately, and choose the pressure to vanish in the shear flow at large y, the boundary condition (2.2) implies that

$$\left.\begin{aligned} \phi(z) &\sim \tfrac{1}{4}z+iU, \\ \pi(z) &\sim -\tfrac{1}{4}z, \end{aligned}\right\} \quad \text{as} \quad y\to\infty. \tag{4.6}$$

The symmetry of the problem requires that

$$[\omega+ip] = 0, \quad [-v+iu] = 0. \tag{4.7}$$

The first condition implies that $[\phi'(z)] = 0$ from (4.2), and (4.6) then implies that

$$[\phi(z)-\tfrac{1}{4}z] = 0. \tag{4.8}$$

The second part of (4.7), with (4.4), requires that $[\phi(z)+\overline{\pi(z)}] = 0$, or

$$[\pi(z)+\tfrac{1}{4}z] = 0. \tag{4.9}$$

Defining $\Phi(\zeta) = \phi(w(\zeta))$ and $\Pi(\zeta) = \pi(w(\zeta))$, the jump conditions (4.8) and (4.9) imply that $\Phi(\zeta)-\frac{1}{4}w(\zeta)$ and $\Pi(\zeta)+\frac{1}{4}w(\zeta)$ are both single-valued analytic functions of ζ in the whole of the region exterior to the unit circle. Morover, from (4.6), $\Phi(\zeta)-\frac{1}{4}w(\zeta)\to iU$ and $\Pi(\zeta)+\frac{1}{4}w(\zeta)\to 0$ as $|\zeta|\to\infty$. It therefore follows that Laurent expansions of the form

$$\left.\begin{aligned}\Phi(\zeta)-\tfrac{1}{4}w(\zeta) &= iU+\sum_{n=1}^{\infty}\frac{ia_n}{\zeta^n},\\ \Pi(\zeta)+\tfrac{1}{4}w(\zeta) &= \sum_{n=1}^{\infty}\frac{ib_n}{\zeta^n}\end{aligned}\right\} \tag{4.10}$$

exist in the exterior of the unit circle, where a_n and b_n are constants.

Given conditions on the bounding surface in the z plane transform to relations between $\Phi(\zeta)$, $\Pi(\zeta)$ and $w(\zeta)$ to be satisfied on the unit circle in the ζ plane. Inserting (4.10), for a given form of $w(\zeta)$, and writing $\zeta = e^{i\theta}$ in them, allows these conditions to be written as the vanishing of particular Fourier series for all θ. We are thus led to a system of equations for a_n and b_n which yields the required solution. If the boundary conditions are symmetric, then both the a_n and b_n are real.

For particular boundary conditions, the above procedure can obviously be simplified, but it has the merit that it can, in principle, be employed whatever conditions are adopted on the undulations. In the following two sections, solutions are obtained for the shear flow (4.6) over the particular surface given by (3.3) with, first, a no-slip condition at the surface and, second, a zero-shear-stress condition.

5. Shear flow over the rough surface with a no-slip condition

From (4.4), the no-slip condition transforms to

$$\Phi(\zeta)+\{w(\zeta)-\overline{w(\zeta)}\}\overline{\Phi'(\zeta)}/\overline{w'(\zeta)}+\overline{\Pi(\zeta)} = 0 \quad \text{on} \quad \zeta = e^{i\theta}. \tag{5.1}$$

Substituting (3.3), (3.4) and (4.10) into this and comparing coefficients, we obtain a system of equations which leads to

$$\left.\begin{aligned}U &= -\alpha^2/2\pi,\\ a_1 &= -\alpha/2\pi; \quad a_n = 0 \quad \text{for} \quad n>1,\\ b_n &= (\alpha^n/2\pi)(1+\alpha^2) \quad \text{for all} \quad n,\end{aligned}\right\} \tag{5.2}$$

so that the solution has the closed form

$$\left.\begin{aligned} w(\zeta) &= \frac{i}{\pi}\log\zeta + \frac{i\alpha}{\pi\zeta},\\ \Phi(\zeta) &= \tfrac{1}{4}w(\zeta) - \frac{i\alpha^2}{2\pi} - \frac{i\alpha}{2\pi\zeta},\\ \Pi(\zeta) &= -\tfrac{1}{4}w(\zeta) + \frac{i\alpha}{2\pi}\frac{1+\alpha^2}{\zeta-\alpha}.\end{aligned}\right\} \tag{5.3}$$

That this solution satisfies (5.1) can now be verified directly. The first relation of (5.2) determines U as a function of α. Having applied a no-slip condition, such a relation is to be expected in the present case. Nevertheless, this example illustrates the method of solution, and the resulting closed form obtained is of interest in other fields involving flows over wavy boundaries. It also offers an interesting comparison with the case considered in the next section.

In fact, the form of (5.1) is such that, when a no-slip condition is relevant, a closed-form solution can be obtained more directly for quite general surface shapes by using (5.1) to continue $\Phi(\zeta)$ analytically into the interior of the unit circle. We do not pursue this here, since the particular case already dealt with suffices for present purposes.

6. Shear flow over the rough surface with a zero-shear-stress condition

The manipulations in this case become somewhat more involved, but are straightforward. We first require that the surface be a streamline, expressed by $\mathrm{Re}\{(-v+iu)\,d\bar{z}\} = 0$, where dz represents an increment along the surface. The zero-shear-stress condition at the boundary is expressed by $\mathrm{Re}\{(F_x+iF_y)\,d\bar{z}\} = 0$. Transforming these conditions to the unit circle in the ζ plane, inserting the representations (4.10) and comparing coefficients, we obtain a system of equations for the a_n and b_n. These may be simplified to yield a second-order difference equation for the coefficients a_n, viz.

$$-\alpha(2n+3)a_{n+2} + 2(\alpha^2 n + n + 1)a_{n+1} - \alpha(2n-1)a_n = \begin{cases}\alpha^2/2\pi & \text{for} \quad n = 1,\\ 0 \quad \text{for} & n > 1,\end{cases} \tag{6.1}$$

to be solved with the initial conditions

$$\left.\begin{aligned} a_1 &= -1/2\alpha\pi,\\ a_2 &= \tfrac{1}{3}U + \frac{1}{2\pi} - \frac{1}{3\alpha^2\pi}.\end{aligned}\right\} \tag{6.2}$$

The b_n are then determined by

$$-\alpha(3n+1)a_{n+1} + (3n-1)a_n - (1+\alpha^2)\sum_{m=1}^{n} m a_m \alpha^{n-m} = \begin{cases} b_1 - \alpha/\pi & \text{for} \quad n = 1,\\ b_n \quad \text{for} & n > 1.\end{cases} \tag{6.3}$$

The difference equation (6.1) and the initial data (6.2) allow each a_n to be expressed in terms of U and α: equation (6.3) then allows all the b_n to be similarly expressed. It would appear, therefore, that the series for $\Phi(\zeta)$ and $\Pi(\zeta)$ can be determined without any restriction on U. However, these series must define functions which are analytic in $|\zeta| \geqslant 1$, and it is this condition which fixes U.

To see this, we note that (6.1) implies that

$$a_{n+2}-(\alpha+1/\alpha)a_{n+1}+a_n \approx 0 \quad \text{for large } n. \tag{6.4}$$

The solution of (6.1) can thus be written as

$$a_n = A(a_1,a_2)f_1(n)+B(a_1,a_2)f_2(n), \tag{6.5}$$

where $A(a_1,a_2)$ and $B(a_1,a_2)$ depend only on the initial values a_1 and a_2, and are thus functions of U and α, while

$$f_1(n) \sim \alpha^n, \quad f_2(n) \sim \alpha^{-n} \quad \text{as} \quad n \to \infty. \tag{6.6}$$

From this, the series for $\Phi(\zeta)$ resulting from $f_1(n)$ converges in $|\zeta| > |\alpha|$, while that from $f_2(n)$ converges only in $|\zeta| > |\alpha|^{-1}$. Since $|\alpha| < 1$ and $\Phi(\zeta)$ is to be regular in $|\zeta| \geqslant 1$ it follows that $B(a_1,a_2) = 0$ This furnishes the required relation giving U in terms of α.

To obtain the analytical form of this relation for U we require the complete solution of the difference equation (6.1), and this may be obtained using a Maclaurin transform. If we define

$$F(\zeta) = \sum_{n=1}^{\infty}\frac{a_n}{\zeta^n} = -i\Phi(\zeta)+\frac{i}{4}w(\zeta)-U, \tag{6.7}$$

then $F(\zeta)$ is to be analytic and single-valued in $|\zeta| \geqslant 1$. If we multiply the nth equation of the system (6.1) by ζ^{-n} and sum over n, we find

$$F'(\zeta)+\frac{1}{2}\left(\frac{1}{\zeta}+\frac{1}{\zeta-1/\alpha}-\frac{1}{\zeta-\alpha}\right)F(\zeta) = -\frac{(a_1\zeta-2a_1/\alpha+3a_2-\alpha/2\pi\zeta)}{2\zeta(\zeta-1/\alpha)(\zeta-\alpha)},$$

which implies that

$$\left\{\frac{\zeta(\zeta-1/\alpha)}{\zeta-\alpha}\right\}^{\frac{1}{2}}F(\zeta) = -\frac{1}{2}\int^{\zeta}\frac{(a_1\zeta-2a_1/\alpha+3a_2-\alpha/2\pi\zeta)}{\zeta^{\frac{1}{2}}(\zeta-1/\alpha)^{\frac{1}{2}}(\zeta-\alpha)^{\frac{3}{2}}}d\zeta. \tag{6.8}$$

To ensure that $F(\zeta)$ is analytic at infinity, and is $O(1/\zeta)$ there, we must begin the integration contour for the integral on the right at infinity. Then, to ensure that $F(\zeta)$ is not singular at $\zeta = 1/\alpha$ this integral must vanish when evaluated at $\zeta = 1/\alpha$, and this restriction gives U as a function of α. Using the initial values a_1 and a_2 given by (6.2), this relation can be reduced to

$$2\pi U = -\frac{\alpha^2E(\alpha)-(1-\alpha^2)K(\alpha)}{E(\alpha)-(1-\alpha^2)K(\alpha)}, \tag{6.9}$$

where $K(\alpha)$ and $E(\alpha)$ are complete elliptic integrals of the first and second kinds, respectively, defined by

$$K(\alpha) = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-\alpha^2\sin^2\theta)^{\frac{1}{2}}}, \quad E(\alpha) = \int_0^{\frac{1}{2}\pi}(1-\alpha^2\sin^2\theta)^{\frac{1}{2}}\,d\theta.$$

These are tabulated as functions of the argument α by Fletcher (1940).

This now furnishes a closed-form solution for the whole flow in terms of elliptic integrals, but our present interest is directed towards the relation (6.9). From this we find that

$$2\pi U \to -1 \quad \text{as} \quad \alpha \to \pm 1, \tag{6.10}$$

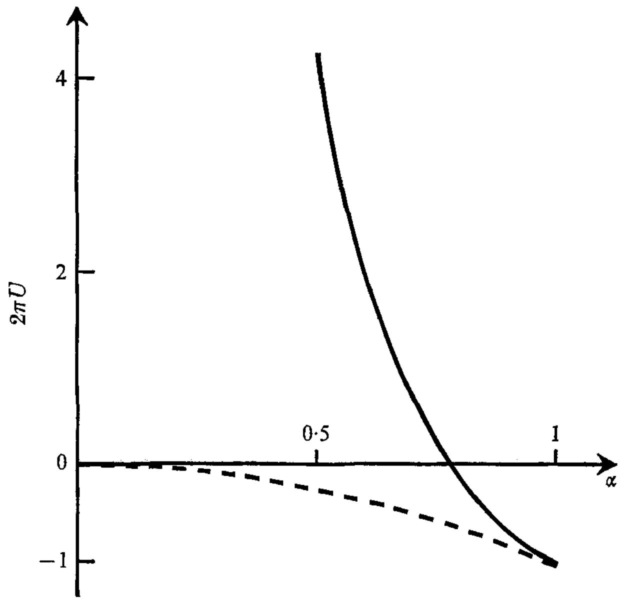

FIGURE 5. The variation of U with α. ---, variation with the no-slip boundary condition, as given in (5.2); ——, variation with the zero-shear-stress boundary condition, as given in (6.9).

while
$$\pi U \sim \alpha^{-2} \quad \text{as} \quad \alpha \to 0, \tag{6.11}$$
in agreement with Nye (1969, 1970). We also have
$$d(\pi U)/d\alpha \to \pm 1 \quad \text{as} \quad \alpha \to \pm 1. \tag{6.12}$$
It should be noted that the limiting values (6.10) and (6.12) for $\alpha \to \pm 1$ are the same as those holding for the relation $2\pi U = -\alpha^2$ which was derived in (5.2) as the relevant form when the no-slip condition is applied: loosely speaking we may say that, for the particular limiting form of the surface with the cusps ($\alpha \to \pm 1$), the zero-shear-stress condition is just as effective as the no-slip condition in producing a macroscopic boundary condition which would be interpreted as no slip. For other forms of the surface with $\alpha \neq 0$, the difference between the two cases is still of the same order of magnitude as the dimensions of the irregularities in the surface, and therefore either of them would lead to a macroscopic no-slip condition. The variation of U with α for the conditions of no slip and zero shear stress analysed here is sketched in figure 5. One might anticipate that the analogous curve obtained by applying some intermediate, and more realistic, condition at the boundary would lie between these two, and hence lead to the same conclusion that the macroscopic no-slip condition must inevitably result.

7. Conclusions and comments

It has been shown in the previous section that a steady shear flow can be maintained over a solid surface at which a zero-shear-stress condition is applied, provided the surface is corrugated. When the surface is plane, no such flow is possible. Regarding this flow as the inner solution obtained near a rough wall, as in § 2, this then implies that, even if there is no resistance at all to a relative motion between a fluid and a solid in contact, the roughness alone will ensure that the boundary condition observed on a macroscopic scale will be one of no slip. It therefore seems reasonable to expect the same conclusion to hold whatever the magnitude of the resistance to relative motion arising from intermolecular forces may be.

The above calculations cannot, of course, be regarded as a proof, but support the contention that the familiar no-slip condition is to be expected because, to use the language of the variational calculus, the surfaces encountered in practice are strong variations from the idealized, smooth surfaces normally envisaged in theoretical work, rather than weak variations, i.e. while the actual and idealized surfaces are close together, their slopes at corresponding points are significantly different. We have here considered only one particular family of surfaces, and considered only the particular extreme boundary condition of perfect slip, in order to obtain a specific problem to be solved. A proof must allow for a general form of roughness and should preferably insist only that the actual boundary condition between the solid and the fluid does not inject energy into the flow. Moreover, the approach used above, invoking matched asymptotic expansions, while suggestive, would probably be so difficult to justify rigorously that a more general attack using techniques of functional analysis is likely to prove simpler.

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